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Reverberation-ray matrix analysis of free vibration of piezoelectric laminates

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ABSTRACT

Based on the theory of first-order ordinary differential equations, a dual relation between two solutions in the dual local coordinates for a single layer in the laminate is derived, which is further arranged in a manner that can avoid the numerical instability usually encountered in the state space method. Joint coupling relation can be established by the consideration of equilibrium and compatibility conditions at the interfaces/surfaces or nodes. The two relations correspond to the phase relation and scattering relation in the method of reverberation-ray matrix (MRRM), as demonstrated by considering the free vibration of orthotropic (or cross-ply) piezoelectric laminates in cylindrical bending. Another contribution of the paper is that the case of repeated eigenvalues of the coefficient matrix of the state equation is discussed, which has never been tackled before. The discussion completes the mathematical formulation of MRRM. The approach is first verified by comparing the results with those obtained from the state space method for laminates with simple supports. Calculations are then performed to show the dominating modes at some given frequencies and particular discussions are presented.

1. Introduction

Intelligent or smart materials have been extensively used in mechanical engineering, civil engineering, navigation, aviation and aerospace for its well adapting to various applied loads and environments. The behavior of such intelligent structures has therefore gained much attention in the mechanics community [1–4]. Exact solutions similar to Pagano's solutions [5,6] have been derived and played an important role in both validating simplified theories and helping understanding the in-depth physics of smart structures [7–11].

Although the finite element method (FEM) has achieved a great success in practical fields, it is not omnipotent since the FEM solutions usually do not provide a direct relation between the results and the physical essence of the problems. Compared to FEM which is now extensively used, the state space method (SSM) [12–15] has several advantages. Firstly, it should be more accurate from the theoretical point of view due to its continuous basis. Secondly, the scale of the final

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equations in SSM does not increase irrespective of the increasing of layer number for layered media, thus making numerical calculation very efficient. However, SSM suffers from serious numerical instabilities for long-element and high-frequency problems when the exponential matrix operation is carried out completely by a computer [16]. Several efficient methods have been proposed to overcome the difficulty [17–20].

Based on the mechanism of wave propagation, Pao and his associates developed a new method, called the method of reverberation-ray matrix (MRRM), to analyze dynamic responses of framed structures [21–23] and transient wave propagation in layered media [24,25]. The formulation bears a strong physical interpretation and can be easily extended to analyze free vibration of piezoelectric laminates as shown immediately. More interestingly, it is found that the MRRM can inherently avoid the aforementioned numerical problem, as recently pointed out by Pao et al. [26].

Although MRRM has shown great feasibility for solving practical problems as well as uniformity in formulations and calculations, its theoretical basis has not been well-established. This paper will provide a mathematically more strict derivation of MRRM based on theory of first-order linear differential equations. In particular, the case of repeated eigenvalues, which has not been considered before by Pao et al., is discussed here to complete the formulating of MRRM. Comparison for simply supported laminates with the conventional SSM is preformed to validate the present MRRM. Finally, normal modes of some selected frequencies are presented to assess the superiority of the present method.

2. Mathematical basis for MRRM

2.1. Solution of ordinary differential equations

Consider the following general set of first-order ordinary differential equations with constant coefficients:

$$\frac{\mathrm{d}\mathbf{f}(z)}{\mathrm{d}z} = \mathbf{C}\mathbf{f}(z) + \mathbf{g}(z) \quad (z_0 \le z \le z_1) \tag{1}$$

where $\mathbf{f}$ is the state vector, whose m elements being unknown functions of z, $\mathbf{C}$ the constant coefficient matrix of order $m \times m$ and $\mathbf{g}$ the prescribed source vector. It should be noted that, any system of ordinary differential equations can be rewritten into the form shown in Eq. (1).

The solution to Eq. (1) is [27]:

$$\mathbf{f}(z) = \mathbf{B}(z, z_0)\mathbf{f}(z_0) + \int_{z_0}^{z} \mathbf{B}(z, \varphi)\mathbf{g}(\varphi)\,\mathrm{d}\varphi \tag{2}$$

where $\mathbf{B}(z, z_0) = \mathrm{e}^{\mathbf{C}(z-z_0)}$ is the transfer matrix or propagator [28].

The solution can also be expressed as

$$\mathbf{f}(z) = \mathrm{e}^{\mathbf{C}z}\left[\mathbf{A} + \int_{z_0}^{z} \mathrm{e}^{-\mathbf{C}\varphi}\mathbf{g}(\varphi)\,\mathrm{d}\varphi\right] \tag{3}$$

where $\mathbf{A} = \mathbf{e}^{-Cz_0}\mathbf{f}(z_0)$, the elements of which can be seen as undetermined constants.

Note that

$$\mathrm{e}^{\mathbf{C}z} = \mathbf{V}\,\mathrm{e}^{\mathbf{\Lambda}z}\,\mathbf{V}^{-1} \tag{4}$$

where $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$ is a diagonal matrix with m elements on the main diagonal and $\mathbf{V} = [\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n]$, with $\mathbf{V}_i$ being the eigenvector corresponding to the eigenvalue λ_i of $\mathbf{C}$. The form of Eq. (4) assumes that all eigenvalues of $\mathbf{C}$ are distinct from each other. We will discuss the case of repeated eigenvalues in a latter stage in Section 5.

Using Eq. (4), Eq. (3) can be further modified as

$$\mathbf{f}(z) = \mathbf{V}\,\mathrm{e}^{\mathbf{\Lambda}z}\left[\mathbf{A}' + \int_{z_0}^{z} \mathrm{e}^{-\mathbf{\Lambda}\varphi}\mathbf{V}^{-1}\mathbf{g}(\varphi)\,\mathrm{d}\varphi\right] \tag{5}$$

where $\mathbf{A}' = \mathbf{V}^{-1}\mathbf{A}$ is another unknown vector.

2.2. Dual coordinate systems and coordinate transform

As a requisite for the MRRM, a pair of dual local coordinates is used for a single layer in a stratified medium or a beam member in a 2D or 3D frame. In the dual local coordinate systems (see Fig. 1), z^{IJ} denotes the system pointing from I to J and *vice versa* for z^{JI}. In the text hereafter, any physical quantity affixed with superscripts IJ or JI denotes that it is associated with the coordinate system IJ or JI, respectively. Note that for any point in the IJ element, there exists $z^{IJ} = h^{IJ} - z^{JI}$, where $h^{IJ} = h^{JI}$ is the length (or depth) of the element.

According to Eq. (3), the solutions of Eq. (1) in the dual local coordinate systems z^{IJ} and z^{JI} are expressed as

$$\mathbf{f}^{IJ}(z^{IJ}) = \mathrm{e}^{\mathbf{C}^{IJ}z^{IJ}}\left[\mathbf{A}^{IJ} + \int_{0}^{z^{IJ}} \mathrm{e}^{-\mathbf{C}^{IJ}\varphi}\mathbf{g}^{IJ}(\varphi)\,\mathrm{d}\varphi\right] \tag{6}$$

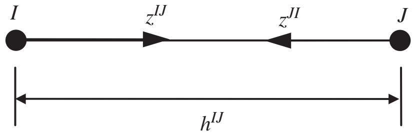

Fig. 1. Dual coordinate systems for a typical layer or element.

$$\mathbf{f}^{JI}(z^{JI}) = \mathrm{e}^{\mathbf{C}^{JI}z^{JI}}\left[\mathbf{A}^{JI} + \int_0^{z^{JI}} \mathrm{e}^{-\mathbf{C}^{JI}\varphi}\mathbf{g}^{JI}(\varphi)\,\mathrm{d}\varphi\right] \tag{7}$$

These solutions are obtained in the dual coordinate systems and may be referred to as dual solutions of the problem.

For a practical mechanics problem, the unknown vector **f** and the inhomogeneous vector **g** usually have definite physical meanings, following a certain transformation rule in different coordinates. Denote the transformation matrix between $\mathbf{f}^{JI}$ and $\mathbf{f}^{IJ}$ and that between $\mathbf{g}^{JI}$ and $\mathbf{g}^{IJ}$ at a same point ($z^{JI} = h^{IJ} - z^{IJ}$) as $\mathbf{T}_f$ and $\mathbf{T}_g$, respectively, i.e.

$$\mathbf{f}^{JI}(h^{IJ} - z^{IJ}) = \mathbf{T}_f\mathbf{f}^{IJ}(z^{IJ}) \tag{8}$$

and

$$\mathbf{g}^{JI}(h^{IJ} - z^{IJ}) = \mathbf{T}_g\mathbf{g}^{IJ}(z^{IJ}) \tag{9}$$

Then, using Eq. (1), we can verify the following equalities:

$$\mathbf{C}^{JI} = -\mathbf{T}_f\mathbf{C}^{IJ}\mathbf{T}_f^{-1}, \quad \mathbf{T}_g = -\mathbf{T}_f \tag{10}$$

and further

$$\mathbf{B}^{IJ}(z^{IJ}, h^{IJ}) = \mathbf{T}_f^{-1}\mathbf{B}^{JI}(h^{IJ} - z^{IJ}, 0)\mathbf{T}_f \tag{11}$$

or

$$\mathrm{e}^{\mathbf{C}^{IJ}(z^{IJ} - h^{IJ})} = \mathbf{T}_f^{-1}\mathrm{e}^{\mathbf{C}^{JI}(h^{IJ} - z^{IJ})}\mathbf{T}_f \tag{12}$$

Eq. (12) also can be derived directly based on the Magnus expansion [29].

2.3. Dual relation

In the following, we will establish a relation, called the dual relation, by noticing the transformation relations presented in Section 2.2.

Considering Eqs. (7)–(9) and (12), one gets

$$\begin{aligned}\mathbf{f}^{IJ}(z^{IJ}) &= \mathbf{T}_f^{-1}\mathbf{f}^{JI}(h^{IJ} - z^{IJ}) = \mathbf{T}_f^{-1}\mathrm{e}^{\mathbf{C}^{JI}(h^{IJ} - z^{IJ})}\left[\mathbf{A}^{JI} + \int_0^{h^{IJ} - z^{IJ}} \mathrm{e}^{-\mathbf{C}^{JI}\varphi}\mathbf{g}^{JI}(\varphi)\,\mathrm{d}\varphi\right] \\ &= \mathrm{e}^{\mathbf{C}^{IJ}(z^{IJ} - h^{IJ})}\mathbf{T}_f^{-1}\left[\mathbf{A}^{JI} - \int_0^{h^{IJ} - z^{IJ}} \mathrm{e}^{-\mathbf{C}^{JI}\varphi}\mathbf{T}_f\mathbf{g}^{IJ}(h^{IJ} - \varphi)\,\mathrm{d}\varphi\right]\end{aligned} \tag{13}$$

By variable substitution, we have

$$\begin{aligned}\mathbf{f}^{IJ}(z^{IJ}) &= \mathrm{e}^{\mathbf{C}^{IJ}(z^{IJ} - h^{IJ})}\mathbf{T}_f^{-1}\left[\mathbf{A}^{JI} + \int_{h^{IJ}}^{z^{IJ}} \mathrm{e}^{-\mathbf{C}^{JI}(h^{IJ} - \varphi)}\mathbf{T}_f\mathbf{g}^{IJ}(\varphi)\,\mathrm{d}\varphi\right] \\ &= \mathrm{e}^{\mathbf{C}^{IJ}z^{IJ}}\left[\mathrm{e}^{-\mathbf{C}^{IJ}h^{IJ}}\mathbf{T}_f^{-1}\mathbf{A}^{JI} + \int_{h^{IJ}}^{z^{IJ}} \mathrm{e}^{-\mathbf{C}^{IJ}\varphi}\mathbf{g}^{IJ}(\varphi)\,\mathrm{d}\varphi\right]\end{aligned} \tag{14}$$

Comparing to Eq. (6), one obtains

$$\begin{aligned}\mathbf{A}^{IJ} &= \mathrm{e}^{-\mathbf{C}^{IJ}h^{IJ}}\mathbf{T}_f^{-1}\mathbf{A}^{JI} - \int_0^{h^{IJ}} \mathrm{e}^{-\mathbf{C}^{IJ}\varphi}\mathbf{g}^{IJ}(\varphi)\,\mathrm{d}\varphi \\ &= \mathbf{V}^{IJ}\mathrm{e}^{-\mathbf{\Lambda}^{IJ}h^{IJ}}(\mathbf{T}_f\mathbf{V}^{IJ})^{-1}\mathbf{A}^{JI} - \mathbf{V}^{IJ}\int_0^{h^{IJ}} \mathrm{e}^{-\mathbf{\Lambda}^{IJ}\varphi}(\mathbf{V}^{IJ})^{-1}\mathbf{g}^{IJ}(\varphi)\,\mathrm{d}\varphi\end{aligned} \tag{15}$$

This equation expresses the dual relation between two groups of unknown constants, which should be suitable for not only dynamic responses, but also static or buckling problems. In addition, for a mechanics problem, the inhomogeneous term in the above formulation corresponds to distributive loads, which have not been considered before [21–25].

Suppose that λ^{IJ} is the eigenvalue of $\mathbf{C}^{IJ}$, viz.

$$(\mathbf{C}^{IJ} - \lambda^{IJ}\mathbf{I})\mathbf{V}_{\lambda}^{IJ} = \mathbf{0} \tag{16}$$

where $\mathbf{V}_{\lambda}^{IJ}$ are the corresponding eigenvector. Left-multiplying the two sides of the above equation with $\mathbf{T}_f$ gives rise to

$$\mathbf{T}_f\mathbf{C}^{IJ}\mathbf{V}_{\lambda}^{IJ} - \lambda^{IJ}\mathbf{T}_f\mathbf{V}_{\lambda}^{IJ} = -(\mathbf{C}^{JI} + \lambda^{IJ}\mathbf{I})\mathbf{T}_f\mathbf{V}_{\lambda}^{IJ} = \mathbf{0} \tag{17}$$

which indicates that, if $(\lambda^{IJ}, \mathbf{V}_{\lambda}^{IJ})$ are the eigenvalue and eigenvector of the matrix $\mathbf{C}^{IJ}$, $(-\lambda^{IJ}, \mathbf{T}_f\mathbf{V}_{\lambda}^{IJ})$ are the counterparts for $\mathbf{C}^{JI}$. Hence, for the eigenvector matrix, we have

$$\mathbf{V}^{JI} = \mathbf{T}_f\mathbf{V}^{IJ} \tag{18}$$

if the eigenvalues are arranged in the following ways:

$$\mathbf{\Lambda}^{IJ} = \mathrm{diag}(\lambda_1^{IJ}, \lambda_2^{IJ}, \ldots, \lambda_m^{IJ})$$

$$\mathbf{\Lambda}^{JI} = \mathrm{diag}(\lambda_1^{JI}, \lambda_2^{JI}, \ldots, \lambda_m^{JI}) = \mathrm{diag}(-\lambda_1^{IJ}, -\lambda_2^{IJ}, \ldots, -\lambda_m^{IJ}) \tag{19}$$

By introducing new unknown coefficients $(\mathbf{A}')^{IJ} = (\mathbf{V}^{IJ})^{-1}\mathbf{A}^{IJ}$ and $(\mathbf{A}')^{JI} = (\mathbf{V}^{JI})^{-1}\mathbf{A}^{JI}$, Eq. (15) can be simplified to

$$(\mathbf{A}')^{IJ} = \mathrm{e}^{-\mathbf{\Lambda}^{IJ}h^{IJ}}(\mathbf{A}')^{JI} - \int_0^{h^{IJ}} \mathrm{e}^{-\mathbf{\Lambda}^{IJ}\zeta}(\zeta)(\mathbf{V}^{IJ})^{-1}\mathbf{g}^{IJ}(\zeta)\,\mathrm{d}\zeta \tag{20}$$

This dual relation will be employed in the following derivation. For the sake of simplicity, we will drop the prime and simply write $\mathbf{A}'$ as $\mathbf{A}$.

2.4. System matrix

The dual relation derived above should be rewritten to assure the stability in numerical calculation. If the real-part of eigenvalue λ^{IJ} of $\mathbf{C}^{IJ}$ is negative, the exponent of the matrix $\mathrm{e}^{-\mathbf{\Lambda}^{IJ}h^{IJ}}$ will contain positive real part. Thus, preserving $\mathbf{A}^{IJ}$ and $\mathbf{A}^{JI}$ to be solved together will inevitably lead to calculation involving large and small numbers synchronously. This usually causes serious instability problem in the numerical calculations. To settle the trouble, it is feasible to suppose that $\mathbf{C}^{IJ}$ has m_0 ($m_0 \le m$) eigenvalues λ_i ($i = 1,2,\ldots,m_0$) with positive real-part or of positive pure imaginary and the rest $m - m_0$ eigenvalues λ_j ($j = m_0+1, m_0+2, \ldots, m$) with negative real-part or of negative pure imaginary. Then we have

$$\mathbf{\Lambda}_+^{IJ} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{m_0}), \quad \mathbf{\Lambda}_-^{IJ} = \mathrm{diag}(\lambda_{m_0+1}, \lambda_{m_0+2}, \ldots, \lambda_n) \tag{21}$$

Correspondingly, the constant vector can be partitioned as

$$\mathbf{A}^{IJ} = \begin{Bmatrix} \mathbf{A}_+^{IJ} \\ \mathbf{A}_-^{IJ} \end{Bmatrix} \tag{22}$$

Thus, $\mathbf{A}_+^{IJ}$ is associated with the solution that exponentially increases or simply oscillates along the coordinate axis IJ and $\mathbf{A}_-^{IJ}$ is with the solution that exponentially decreases or simply oscillates along that coordinate. Owing to Eq. (19), $\mathbf{A}^{JI}$ can be decomposed similarly as

$$\mathbf{A}^{JI} = \begin{Bmatrix} \mathbf{A}_-^{JI} \\ \mathbf{A}_+^{JI} \end{Bmatrix} \tag{23}$$

where $\mathbf{A}_-^{JI}$ is associated with the solution that exponentially decreases or simply oscillates along the coordinate axis JI and $\mathbf{A}_+^{JI}$ corresponds to that exponentially increases or simply oscillates. By further assuming

$$-\int_0^{h^{IJ}} \mathrm{e}^{-\mathbf{\Lambda}^{IJ}\varphi}(\mathbf{V}^{IJ})^{-1}\mathbf{g}^{IJ}(\varphi)\,\mathrm{d}\varphi = \begin{Bmatrix} \mathbf{Q}_1^{IJ} \\ \mathbf{Q}_2^{IJ} \end{Bmatrix} \tag{24}$$

Eq. (20) can be rewritten as

$$\mathbf{A}_+^{IJ} = \mathrm{e}^{-\mathbf{\Lambda}_+^{IJ}h^{IJ}}\mathbf{A}_-^{JI} + \mathbf{Q}_1^{IJ}, \quad \mathbf{A}_-^{IJ} = \mathrm{e}^{-\mathbf{\Lambda}_-^{IJ}h^{IJ}}\mathbf{A}_+^{JI} + \mathbf{Q}_2^{IJ} \tag{25}$$

which can be further rearranged as

$$\begin{Bmatrix} \mathbf{A}_+^{IJ} \\ \mathbf{A}_+^{JI} \end{Bmatrix} = \begin{bmatrix} \mathbf{0} & \mathrm{e}^{-\mathbf{\Lambda}_+^{IJ}h^{IJ}} \\ \mathrm{e}^{\mathbf{\Lambda}_-^{IJ}h^{IJ}} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{A}_-^{IJ} \\ \mathbf{A}_-^{JI} \end{Bmatrix} + \begin{Bmatrix} \mathbf{Q}_1^{IJ} \\ -\mathrm{e}^{\mathbf{\Lambda}_-^{IJ}h^{IJ}}\mathbf{Q}_2^{IJ} \end{Bmatrix} \tag{26}$$

The above equation can be written in a more compact form as follows:

$$\mathbf{a}^{IJ} = \mathbf{P}^{IJ}\mathbf{d}^{IJ} + \mathbf{s}^{IJ} \tag{27}$$

which, according to the previous works on MRRM [26], can be termed as the generalized local phase relation. An obvious difference may be observed that the phase relation obtained before is homogeneous, while Eq. (27) is inhomogeneous due to the involvement of inhomogeneous term in Eq. (1). By properly dividing the unknown constants into two groups, large numbers are avoided in the phase matrix $\mathbf{P}^{IJ}$, which assures numerical stability in the calculation. Therefore, MRRM is superior to SSM when dealing with problems with long elements or in the high-frequency range.

With only the phase relation known for every layer, it is still insufficient for the global analysis. Some additional equations at nodes or boundaries (termed here as the joint coupling relation; in previous works on MRRM [26], it is known as the scattering relation) should be established. For a mechanics problem, this relation can always be deduced from the equilibrium and compatibility conditions at the nodes or boundaries (i.e. at the origin of all local coordinates). With $z^{JI} = z^{JK} = 0$, the joint coupling relation contains no exponential functions and hence completely eliminating large numbers. This clearly indicates that formulating of the joint coupling relation is not crucial to the numerical stability induced by large numbers. Hence, the coupling relation at joint J can be arranged according to the previous division of unknown constants as

$$\mathbf{d}^{J} = \mathbf{S}^{J}\mathbf{a}^{J} + \mathbf{Q}^{J} \tag{28}$$

where $\mathbf{S}^{J}$ is termed here the local coupling matrix, $\mathbf{Q}^{J}$ corresponds to external stimuli at the joint. The detailed form for free vibration problem of piezoelectric laminates will be given in the next section.

Assembling all generalized local phase relations for members and all joint coupling relations at nodes, yields

$$\mathbf{a} = \mathbf{P}\mathbf{d} + \mathbf{s}_1 \tag{29}$$

$$\bar{\mathbf{d}} = \mathbf{S}\bar{\mathbf{a}} + \mathbf{s}_2 \tag{30}$$

where $\mathbf{P}$ and $\mathbf{S}$ are the global phase and joint coupling matrices, $\mathbf{s}_1$ and $\mathbf{s}_2$ are the global source vectors due to stimuli applied on members and joints, respectively. The elements of the two vectors $\mathbf{d}$ and $\bar{\mathbf{d}}$ or $\mathbf{a}$ and $\bar{\mathbf{a}}$ are the same, but their sequences may be different. To account for this difference, we introduce the permutation matrices $\mathbf{U}_1$ and $\mathbf{U}_2$ so that

$$\mathbf{d} = \mathbf{U}_1\bar{\mathbf{d}}, \quad \bar{\mathbf{a}} = \mathbf{U}_2\mathbf{a} \tag{31}$$

Combining Eqs. (29) and (31) gives

$$\mathbf{a} = \mathbf{P}\mathbf{U}_1\bar{\mathbf{d}} + \mathbf{s}_1 \tag{32}$$

Further combining Eqs. (30)–(32) yields

$$\bar{\mathbf{d}} = \mathbf{S}\mathbf{U}_2\mathbf{P}\mathbf{U}_1\bar{\mathbf{d}} + \mathbf{S}\mathbf{U}_2\mathbf{s}_1 + \mathbf{s}_2 \tag{33}$$

or briefly,

$$(\mathbf{I} - \mathbf{R})\bar{\mathbf{d}} = \mathbf{s} \tag{34}$$

where $\mathbf{R} = \mathbf{S}\mathbf{U}_2\mathbf{P}\mathbf{U}_1$ is termed here as the system matrix and $\mathbf{s} = \mathbf{S}\mathbf{U}_2\mathbf{s}_1 + \mathbf{s}_2$ is the global source vector due to all external stimuli. In elastodynamic problems, the system matrix is called the reverberation-ray matrix, which has a strong physical interpretation of wave propagation in structures or layered media [21–26]. Note that the expression for the system matrix here seems to be more general than the reverberation-ray matrix and is suitable for any linear system described by Eq. (1).

When the global source vector vanishes in Eq. (34) and for a nontrivial solution, the determinant of the coefficient matrix should vanish, i.e.

$$|\mathbf{I} - \mathbf{R}| = 0 \tag{35}$$

This is the characteristic equation of the linear system. For the free vibration of a structure, Eq. (35) corresponds to the frequency equation, while for the wave propagation in stratified medium, it gives the dispersion relation.

3. State equation of a piezoelectric layer

Now we turn to consider the free vibration of an N-layered cross-ply piezoelectric laminate subjected to cylindrical bending, as shown in Fig. 2. In this case, the two non-zero displacement components are u and w in x and z directions, respectively.

The constitutive relations for an orthotropic piezoelectric layer under cylindrical bending are

$$\sigma_x = c_{11}\frac{\partial u}{\partial x} + c_{13}\frac{\partial w}{\partial z} + e_{31}\frac{\partial \phi}{\partial z}, \quad \sigma_y = c_{12}\frac{\partial u}{\partial x} + c_{23}\frac{\partial w}{\partial z} + e_{32}\frac{\partial \phi}{\partial z}$$

$$\sigma_z = c_{13}\frac{\partial u}{\partial x} + c_{33}\frac{\partial w}{\partial z} + e_{33}\frac{\partial \phi}{\partial z}, \quad \tau_{xz} = c_{55}\frac{\partial u}{\partial z} + c_{55}\frac{\partial w}{\partial x} + e_{15}\frac{\partial \phi}{\partial x}$$

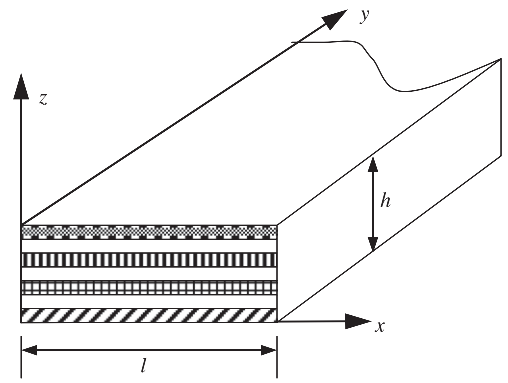

Fig. 2. Cross-ply piezoelectric laminate subjected to cylindrical bending.

$$D_x = e_{15}\frac{\partial u}{\partial z} + e_{15}\frac{\partial w}{\partial x} - \varepsilon_{11}\frac{\partial \phi}{\partial x}, \quad D_z = e_{31}\frac{\partial u}{\partial x} + e_{33}\frac{\partial w}{\partial z} - \varepsilon_{33}\frac{\partial \phi}{\partial z} \tag{36}$$

where c_{ij}, e_{ij} and ε_{ij} are, respectively, the elastic, piezoelectric and dielectric constants, σ_{ij} (τ_{ij}), D_i and ϕ are the normal (shear) stresses, electric displacements and electric potential, respectively. The equations of motion and conservation of charge are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = \rho\frac{\partial^2 u}{\partial t^2}, \quad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_z}{\partial z} = \rho\frac{\partial^2 w}{\partial t^2}, \quad \frac{\partial D_x}{\partial x} + \frac{\partial D_z}{\partial z} = 0 \tag{37}$$

From Eqs. (36) and (37), it is easy to obtain the following state equation [3,15]:

$$\frac{\partial}{\partial z}[u, \sigma_z, D_z, \tau_{xz}, w, \phi]^{\mathrm{T}} = \mathbf{K}[u, \sigma_z, D_z, \tau_{xz}, w, \phi]^{\mathrm{T}} \tag{38}$$

where $\mathbf{K}$ is an operator matrix that is given in Appendix A, u, σ_z, D_z, τ_{xz}, w and ϕ are six state variables. The two so-called induced variables, σ_x and D_x, can be expressed in terms of the state variables, see Eq. (A.2) in Appendix A.

In this paper, the laminate is assumed to be simply supported at the two edges ($x = 0$ and l), which satisfy the following boundary conditions:

$$\sigma_x = w = \phi = 0 \quad (x = 0 \text{ or } l) \tag{39}$$

Compared with that for an elastic plate [30], here the electric potential is also required to be zero at the two edges. For the boundary conditions in Eq. (39), we assume

$$\begin{Bmatrix} u \\ \sigma_z \\ D_z \\ \tau_{xz} \\ w \\ \phi \end{Bmatrix} = \begin{Bmatrix} h\bar{u}(\zeta)\cos n\pi\xi \\ -c_{55}^{(1)}\bar{\sigma}_z(\zeta)\sin n\pi\xi \\ -\sqrt{c_{55}^{(1)}\varepsilon_{33}^{(1)}}\bar{D}_z(\zeta)\sin n\pi\xi \\ c_{55}^{(1)}\bar{\tau}_{xz}(\zeta)\cos n\pi\xi \\ h\bar{w}(\zeta)\sin n\pi\xi \\ h\sqrt{c_{55}^{(1)}/\varepsilon_{33}^{(1)}}\bar{\phi}(\zeta)\sin n\pi\xi \end{Bmatrix} \mathrm{e}^{\mathrm{i}\omega t} \tag{40}$$

where $\zeta = z/h$ and $\xi = x/l$ are the dimensionless coordinates, n the integer (half-wave number), ω the circular frequency and $c_{55}^{(1)}$ and $\varepsilon_{33}^{(1)}$ represent the material constants of the first layer (the bottom layer, see Fig. 2).

Substitution of Eq. (40) into Eq. (38) yields

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\mathbf{f}(\zeta) = \mathbf{C}\mathbf{f}(\zeta) \tag{41}$$

where $\mathbf{f}(\zeta) = [\bar{u}, \bar{\sigma}_z, \bar{D}_z, \bar{\tau}_{xz}, \bar{w}, \bar{\phi}]^{\mathrm{T}}$ is the dimensionless state vector and the expression of the coefficient matrix $\mathbf{C}$ is given in Appendix A.

4. Free vibration of piezoelectric laminates

In this section, we shall follow the derivation in Section 2 to establish the formulation of MRRM for the problem considered in the last section. We assume that the top and bottom surfaces are mechanically tractions-free and electrically open-circuited. Then we have the following boundary conditions:

$$\sigma_z = D_z = \tau_{xz} = 0, \quad \text{at } z = 0, h \tag{42}$$

In addition, the bonding between two adjacent layers is perfect, so the following compatibility conditions hold:

$$\mathbf{f}^{JK}(0) = \mathrm{diag}(-1, 1, -1, 1, -1, 1) \cdot \mathbf{f}^{JI}(0) \triangleq \mathbf{T_f} \cdot \mathbf{f}^{JI}(0) \quad \text{(at the interface } J) \tag{43}$$

From Eq. (42), one gets

$$[\mathbf{V}_+^{01} | \mathbf{V}_-^{01}]_{3\times6} \begin{bmatrix} \mathbf{A}_+^{01} \\ \mathbf{A}_-^{01} \end{bmatrix}_{6\times1} = \mathbf{0}_{3\times1}, \quad [\mathbf{V}_-^{N(N-1)} | \mathbf{V}_+^{N(N-1)}]_{3\times6} \begin{bmatrix} \mathbf{A}_-^{N(N-1)} \\ \mathbf{A}_+^{N(N-1)} \end{bmatrix}_{6\times1} = \mathbf{0}_{3\times1} \tag{44}$$

where $[\mathbf{V}_+^{01} | \mathbf{V}_-^{01}]$ and $[\mathbf{V}_-^{N(N-1)} | \mathbf{V}_+^{N(N-1)}]$ comprise of the second, third and fourth rows of $\mathbf{V}^{01}$ and $\mathbf{V}^{N(N-1)}$, respectively. Eq. (44) can be further written as

$$\mathbf{V}_-^{01}\mathbf{A}_-^{01} = -\mathbf{V}_+^{01}\mathbf{A}_+^{01}, \quad \mathbf{V}_-^{N(N-1)}\mathbf{A}_-^{N(N-1)} = -\mathbf{V}_+^{N(N-1)}\mathbf{A}_+^{N(N-1)} \tag{45}$$

or

$$\mathbf{D}^1\mathbf{d}^1 = \mathbf{S}^1\mathbf{a}^1, \quad \mathbf{D}^N\mathbf{d}^N = \mathbf{S}^N\mathbf{a}^N \tag{46}$$

Here, in contrast to the joint coupling equation presented in Section 2, we keep the matrices $\mathbf{D}^1$ and $\mathbf{D}^N$ on the left-hand sides so that matrix inversion is avoided, which can further assure numerical stability in some extreme cases [31].

On the other hand, Eq. (43) implies

$$\mathbf{V}^{JK}\mathbf{A}^{JK} = \mathbf{T}_\mathrm{f} \cdot \mathbf{V}^{JI}\mathbf{A}^{JI} \tag{47}$$

which is equivalent to

$$\mathbf{V}^{JK}\mathbf{A}^{JK} = \mathbf{V}^{IJ}\mathbf{A}^{JI} \tag{48}$$

This equation can be rearranged as

$$\begin{bmatrix} -\mathbf{V}_{11}^{IJ} & \mathbf{V}_{12}^{JK} \\ -\mathbf{V}_{21}^{IJ} & \mathbf{V}_{22}^{JK} \end{bmatrix} \begin{Bmatrix} \mathbf{A}_-^{JI} \\ \mathbf{A}_-^{JK} \end{Bmatrix} = \begin{bmatrix} \mathbf{V}_{12}^{IJ} & -\mathbf{V}_{11}^{JK} \\ \mathbf{V}_{22}^{IJ} & -\mathbf{V}_{21}^{JK} \end{bmatrix} \begin{Bmatrix} \mathbf{A}_+^{JI} \\ \mathbf{A}_+^{JK} \end{Bmatrix}$$

or

$$\mathbf{D}^J\mathbf{d}^J = \mathbf{S}^J\mathbf{a}^J \tag{49}$$

where $J = 2, 3, \ldots, N-1$. As before, the inversion of matrix $\mathbf{D}^J$ in Eq. (49) is not performed.

Combining Eq. (49) with Eq. (46) gives the global joint coupling relation

$$\begin{bmatrix} \mathbf{D}^1 & \mathbf{0}_{3\times6} & \cdots & \mathbf{0}_{3\times6} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{6\times3} & \mathbf{D}^2 & \vdots & \mathbf{0}_{6\times6} & \mathbf{0}_{6\times3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{6\times3} & \mathbf{0}_{6\times6} & \cdots & \mathbf{D}^{N-1} & \mathbf{0}_{6\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times6} & \cdots & \mathbf{0}_{3\times6} & \mathbf{D}^N \end{bmatrix} \begin{Bmatrix} \mathbf{A}_-^{01} \\ \mathbf{A}_-^{10} \\ \vdots \\ \mathbf{A}_-^{(N-1)N} \\ \mathbf{A}_-^{N(N-1)} \end{Bmatrix} = \begin{bmatrix} \mathbf{A}^1 & \mathbf{0}_{3\times6} & \cdots & \mathbf{0}_{3\times6} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{6\times3} & \mathbf{A}^2 & \vdots & \mathbf{0}_{6\times6} & \mathbf{0}_{6\times3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{6\times3} & \mathbf{0}_{6\times6} & \cdots & \mathbf{A}^{N-1} & \mathbf{0}_{6\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times6} & \cdots & \mathbf{0}_{3\times6} & \mathbf{A}^N \end{bmatrix} \begin{Bmatrix} \mathbf{A}_+^{01} \\ \mathbf{A}_+^{10} \\ \vdots \\ \mathbf{A}_+^{(N-1)N} \\ \mathbf{A}_+^{N(N-1)} \end{Bmatrix}$$

or

$$\mathbf{Dd} = \mathbf{Sa} \tag{50}$$

In the free vibration problem, Eq. (50) is also referred to as the global scattering relation.

The local phase relation of the problem has the same form as Eq. (26). Hence, by assembling such relations for all layers, we obtain the global phase relation

$$\begin{Bmatrix} \mathbf{A}_+^{01} \\ \mathbf{A}_+^{10} \\ \vdots \\ \mathbf{A}_+^{(N-1)N} \\ \mathbf{A}_+^{N(N-1)} \end{Bmatrix} = \begin{bmatrix} \mathbf{P}_{6\times6}^{01} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{P}_{6\times6}^{(N-1)N} \end{bmatrix} \begin{Bmatrix} \mathbf{A}_-^{01} \\ \mathbf{A}_-^{10} \\ \vdots \\ \mathbf{A}_-^{(N-1)N} \\ \mathbf{A}_-^{N(N-1)} \end{Bmatrix}$$

or

$$\mathbf{a} = \mathbf{Pd} \tag{51}$$

It is seen that there is no need to introduce the permutation matrices and the characteristic equation for the problem is

$$|\mathbf{D} - \mathbf{AP}| = 0 \tag{52}$$

As mentioned before, the above formulation of MRRM is valid for the case that all eigenvalues of the coefficient matrix **C** are distinct from each other. We will consider the case of repeated eigenvalues that may appear in the computation and has not been considered before.

5. Case of repeated eigenvalues

When the coefficient matrix **C** has repeated eigenvalues, the solution to Eq. (41) no longer bears the form of $\mathbf{f}(z) = e^{\mathbf{C}z}\mathbf{A}$. For the present problem, we find in the calculation that there may appear two identical zero eigenvalues. Hence, we will discuss the particular case of two identical eigenvalues in the following and the more complicated cases can be treated in a similar way.

In the case that there are two repeated eigenvalues, denoted as λ^*, the expression for the displacement $\bar{u}$ shall take the following form:

$$\bar{u} = A_1\,e^{\lambda_1 z} + A_2\,e^{\lambda_2 z} + A_3\,e^{\lambda_3 z} + A_4\,e^{\lambda_4 z} + A_5 z\,e^{\lambda^* z} + A_6(h - z)\,e^{\lambda^* z} \tag{53}$$

where without loss of generality, we assume that λ_1 and λ_2 represent the exponentially growing terms of the solution and λ_3, λ_4 denote the exponentially decreasing ones. Note that for a practical mechanics problem, the numbers of growing and decreasing terms are always identical.

In the dual local coordinates, we have

$$\bar{u}^{IJ} = A_1^{IJ}\,e^{\lambda_1 z^{IJ}} + A_2^{IJ}\,e^{\lambda_2 z^{IJ}} + A_3^{IJ}\,e^{\lambda_3 z^{IJ}} + A_4^{IJ}\,e^{\lambda_4 z^{IJ}} + A_5^{IJ} z^{IJ}\,e^{\lambda^* z^{IJ}} + A_6^{IJ}(h - z^{IJ})\,e^{\lambda^* z^{IJ}} \tag{54}$$

$$\bar{u}^{JI} = A_1^{JI}\,e^{-\lambda_1 z^{JI}} + A_2^{JI}\,e^{-\lambda_2 z^{JI}} + A_3^{JI}\,e^{-\lambda_3 z^{JI}} + A_4^{JI}\,e^{-\lambda_4 z^{JI}} + A_6^{JI} z^{JI}\,e^{-\lambda^* z^{JI}} + A_5^{JI}(h - z^{JI})\,e^{-\lambda^* z^{JI}} \tag{55}$$

Noting that at the same point, the two expressions shall predict the same result, i.e.

$$\bar{u}^{IJ}(z^{IJ}) = -\bar{u}^{JI}(h^{IJ} - z^{IJ}) \tag{56}$$

one gets

$$\begin{Bmatrix} A_1^{IJ} \\ A_2^{IJ} \\ A_3^{IJ} \\ A_4^{IJ} \\ A_5^{IJ} \\ A_6^{IJ} \end{Bmatrix} = \begin{bmatrix} -e^{-\lambda_1 h^{IJ}} & & & & & \\ & -e^{-\lambda_2 h^{IJ}} & & & & \\ & & -e^{-\lambda_3 h^{IJ}} & & & \\ & & & -e^{-\lambda_4 h^{IJ}} & & \\ & & & & -e^{-\lambda^* h^{IJ}} & \\ & & & & & -e^{-\lambda^* h^{IJ}} \end{bmatrix} \begin{Bmatrix} A_1^{JI} \\ A_2^{JI} \\ A_3^{JI} \\ A_4^{JI} \\ A_5^{JI} \\ A_6^{JI} \end{Bmatrix} \tag{57}$$

which is the dual relation between the unknown constants in the dual local coordinate systems for this peculiar case. To avoid numerical instability, the dual relation can be rearranged to obtain the following local phase relation:

$$\begin{Bmatrix} A_1^{IJ} \\ A_2^{IJ} \\ A_5^{IJ} \\ A_3^{JI} \\ A_4^{JI} \\ A_6^{JI} \end{Bmatrix} = \begin{bmatrix} & & & -e^{-\lambda_1 h^{IJ}} & & \\ & & & & -e^{-\lambda_2 h^{IJ}} & \\ & & & & & -e^{-\lambda^* h^{IJ}} \\ -e^{\lambda_3 h^{IJ}} & & & & & \\ & -e^{\lambda_4 h^{IJ}} & & & & \\ & & -e^{\lambda^* h^{IJ}} & & & \end{bmatrix} \begin{Bmatrix} A_3^{IJ} \\ A_4^{IJ} \\ A_6^{IJ} \\ A_1^{JI} \\ A_2^{JI} \\ A_5^{JI} \end{Bmatrix} \tag{58}$$

As mentioned earlier, only $\lambda^* = 0$ has been encountered in our numerical calculations. Thus there is no numerical instability associated with the term $e^{\lambda^* h^{IJ}}$ in Eq. (58).

In accordance with the expression for $\bar{u}$, the one for the state vector can be expressed as

$$\mathbf{f} = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{16} \\ \gamma_{21} & \cdots & \gamma_{26} \\ \vdots & \vdots & \vdots \\ \gamma_{61} & \cdots & \gamma_{66} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 z} & & & & & \\ & e^{\lambda_2 z} & & & & \\ & & e^{\lambda_3 z} & & & \\ & & & e^{\lambda_4 z} & & \\ & & & & z\,e^{\lambda^* z} & \\ & & & & & (h-z)\,e^{\lambda^* z} \end{bmatrix} \mathbf{A} \tag{59}$$

where $\mathbf{A} = [A_1\ \ A_2\ \ A_3\ \ A_4\ \ A_5\ \ A_6]^{\mathrm{T}}$ for the IJ coordinates, while $\mathbf{A} = [A_1\ \ A_2\ \ A_3\ \ A_4\ \ A_6\ \ A_5\ \ ]^{\mathrm{T}}$ for the JI coordinates and γ_{ij} are constants. Obviously, we have $\gamma_{1i} = 1$ ($i = 1,2,\ldots,6$), while the others can be determined from Eqs. (41) and (58).

At the interface, we have $z = 0$, hence

$$\mathbf{f}(0) = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{14} & 0 & h\gamma_{16} \\ \gamma_{21} & \cdots & \gamma_{24} & 0 & h\gamma_{26} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_{61} & \cdots & \gamma_{64} & 0 & h\gamma_{66} \end{bmatrix} \mathbf{A} \triangleq \overline{\mathbf{V}}\mathbf{A} \tag{60}$$

which shall be used in the derivation of scattering relations, omitted here for brevity.

Table 1
Material properties of piezoelectric materials.

Material	c_{11} (GPa)	c_{33} (GPa)	c_{13} (GPa)	c_{55} (GPa)	e_{15} (c/m^2)	e_{31} (c/m^2)	e_{33} (c/m^2)	$\varepsilon_{11}/\varepsilon_0$	$\varepsilon_{33}/\varepsilon_0$
PZT-4	139	115	74.3	25.6	12.7	−5.2	15.1	1475	1300
PVDF	238.24	10.64	2.19	4.4	−0.135	−0.13	−0.276	12.5	11.98

Note: $\varepsilon_0 = 8.854185 \times 10^{-12}$ (F/m).

Table 2
Lowest frequency parameters (Ω) of a four-ply laminate ([$Z/V/Z/V$], $s = 0.1$).

Mode n	SSM	RMM	Mode n	SSM	RMM	Mode n	SSM	RMM
1	0.0404357	0.04043570	31	4.36837171	4.36837172	51	5.696719	5.69666000
2	0.13627286	0.13627286	32	4.43546978	4.43546978	52	5.76689382	5.76691554
3	0.2560421	0.25604210	33	4.5014792	4.50147921	53	5.83762466	5.8376345
4	0.38847343	0.38847343	34	4.56674669	4.56674668	54	5.90881877	5.90880906
5	0.53130601	0.53130601	35	4.63154735	4.63154735	55	5.98051888	5.98043059
6	0.68431323	0.68431323	36	4.69609932	4.69609933	56	–	6.05248985
7	0.84744268	0.84744268	37	4.76057518	4.76057521	57	–	6.12497720
8	1.02042808	1.02042808	38	4.8251112	4.82511124	58	–	6.19788268
9	1.20271094	1.20271094	39	4.88981456	4.88981462	59	–	6.27119616
10	1.39341523	1.39341523	40	4.95476891	4.95476922	60	–	6.34490740

Note: '–' indicates result not available or presented.

Table 3
Lowest frequency parameters (Ω) of a three-ply hybrid laminate ([$Z/E/V$], $s = 0.15$).

Mode n	SSM	RMM	Mode n	SSM	RMM	Mode n	SSM	RMM
1	0.08300497	0.08300497	31	4.46849245	4.46849105	41	5.72699999	5.73036129
2	0.28550561	0.28550561	32	4.59228664	4.5922838	42	5.849	5.85886169
3	0.55599967	0.55599967	33	4.71671814	4.71671717	43	5.95599998	5.98767794
4	0.86773161	0.86773161	34	4.84175084	4.84174379	44	–	6.11679048
5	1.19275263	1.19275263	35	4.96732059	4.96730078	45	–	6.2461813
6	1.49363742	1.49363742	36	5.09342409	5.09340835	46	–	6.37583376
7	1.73480639	1.73480639	37	5.22026563	5.21997127	47	–	6.50573247
8	1.91018096	1.91018096	38	5.346	5.34697668	48	–	6.63586318
9	2.04133352	2.04133352	39	5.47349988	5.4743947	49	–	6.7662127
10	2.15018215	2.15018215	40	5.60153993	5.60219794	50	–	6.89676875

Note: 'E' represents an elastic layer which has the same elastic constants and mass density as PZT.

6. Numerical examples

In the numerical examples, we assume that the piezoelectric laminates are made of PZT-4 and PVDF, whose material constants [32] are listed in Table 1.

For all examples, except the second example (Table 3), each ply involved in the N-ply laminate is assumed to have the same thickness, i.e. h/N. The orientation angle between the x-axis and material principal axis is denoted as θ, with the assumption that $\theta = 0^\circ$ for the PZT-4 layer and $\theta = 90^\circ$ for the PVDF layer. The transformation of material constants between different Cartesian coordinates can be found in Appendix A of Ref. [33]. Similar to the work by Heyliger and Saravanos [34], the densities of PZT-4 and PVDF herein are assumed the same. In the following, the mechanical boundary conditions at $x = 0$ and $x = l$ are always simply supported and electrically close-circuited, while at the top and bottom surfaces are assumed to be tractions-free and electrically open-circuited.

The MRRM and SSM results for a four-ply laminate ($Z/V/Z/V$, Z: PZT-4 and V: PVDF) with the thickness-to-span ratio $s = h/l = 0.1$ are given in Table 2. It can be seen that, when n is less than 10, the dimensionless frequencies obtained by the SSM agree well with MRRM's results; when n is between 31 and 50, the frequencies by SSM deviate to some degree from those by the MRRM; while n is larger than 56, the SSM encounters serious numerical instability and no results can be obtained. In contrast to SSM, the present method overcomes this trouble perfectly and hence can be an appropriate choice for the high-frequency analysis. The second example considers the free vibration of a hybrid laminate with a layup [$Z/E/V$], here E denotes an elastic ply with the same elastic properties and mass density as PZT. The thicknesses of three plies are non-equal and taken as $h^{12} = h^{34} = 10h^{23}$. The results are given in Table 3 for $s = h/l = 0.15$. Similar to Table 2, we can see

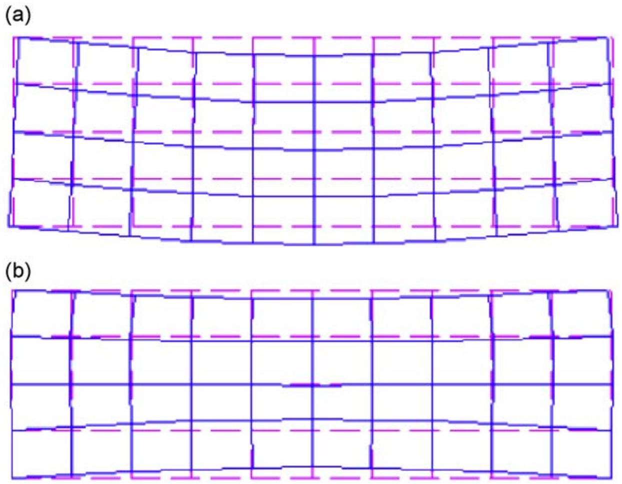

Fig. 3. Normalized mode of the laminate [$Z/V/Z/V$] for $s = 0.1$, $n = 1$: (a) $\Omega = 0.04043570$ and (b) $\Omega = 2.34786915$.

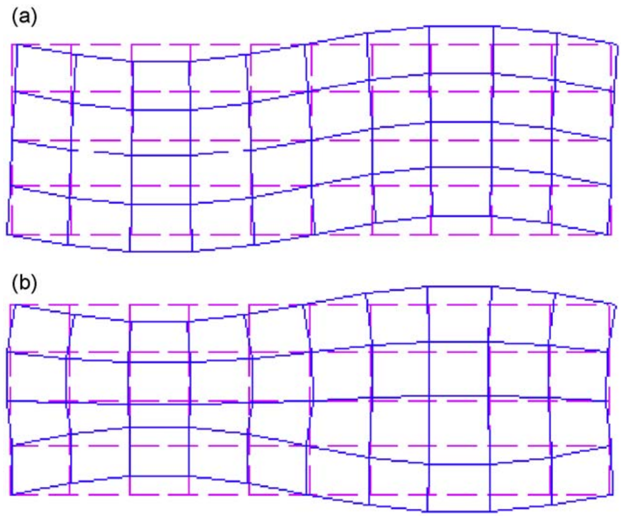

Fig. 4. Normalized mode of the laminate [$Z/V/Z/V$] for $s = 0.1$, $n = 2$: (a) $\Omega = 0.13627286$ and (b) $\Omega = 2.44862178$.

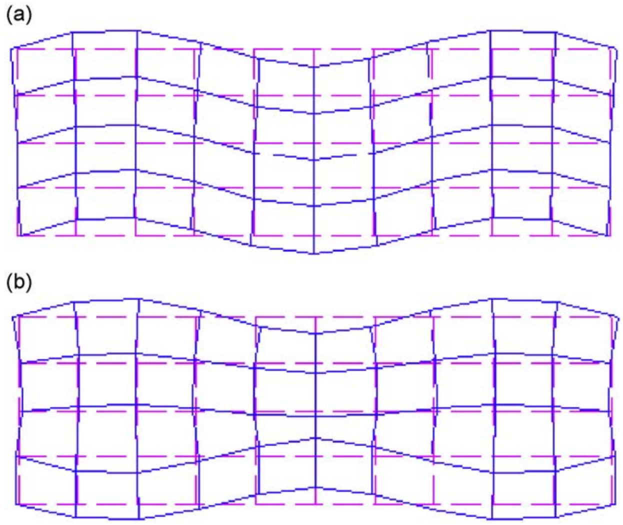

Fig. 5. Normalized mode of the laminate $[Z/V/Z/V]$ for $s = 0.1$, $n = 3$: (a) $\Omega = 0.2560421$ and (b) $\Omega = 2.57501399$.

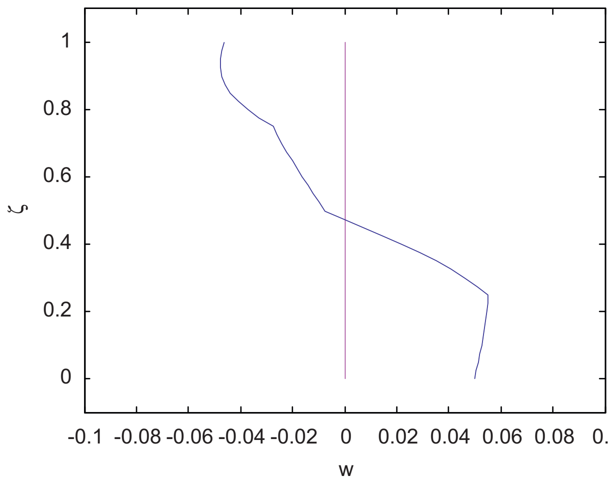

Fig. 6. Distribution of w along the thickness of the laminate $[Z/V/Z/V]$ for $s = 0.1$, $n = 1$ at $\Omega = 2.34786915$.

that when n is larger than 43, the SSM suffers from a serious numerical problem and cannot predict the proper frequency, while the MRRM still works very well.

It is pointed out that, in the first example of the $[Z/V/Z/V]$ laminate, when $\Omega = 0.30163789$ for $n = 1$, as well as $\Omega = 1.46408835$ for $n = 2$, etc., the case of repeated eigenvalues considered in Section 5 is encountered during the search of characteristic roots of system equation, which will give spurious solution if using the formulation for distinct eigenvalues only. Thus, it is very important to consider the case of repeated eigenvalues in the formulation of MRRM, as described in Section 5.

The following discussions are confined to four-ply piezoelectric laminates with different layups. The normalized modes of the $[Z/V/Z/V]$ laminate shown in Figs. 3–5 clearly indicate the dominating deformation for a particular vibrating frequency. Figs. 3a and b show the flexure dominated mode and thickness-stretch dominated mode for $n = 1$, respectively, while those for $n = 2$ and 3 are given in Figs. 4a and b and 5a and b, respectively. For simplicity, we will drop the word 'dominated' assuming that no confusion will be caused. The thickness distribution of transverse displacement w for the thickness-stretch mode in Fig. 3b is shown in Fig. 6. As can be seen, w gains its maximum at the top or bottom surface, while a zero value point is near the middle surface. This is a typical characteristic of the fundamental thickness-stretch mode of a plate structure [35]. Since an asymmetric laminated structure is considered here, the fundamental thickness-stretch mode is just nearly symmetric (not strictly) with respect to the middle surface. At $\Omega = 5.29024761$ and 7.29328146, we have the second and third thickness-stretch modes as shown in Figs. 7 and 8, respectively. For the second thickness stretch mode, the transverse displacement has three maxima along the thickness and is nearly antisymmetrically distributed, as clearly seen from Fig. 7b. The third thickness-stretch mode is again nearly symmetric with respect to the middle surface and the vertical displacement has four maxima along the thickness.

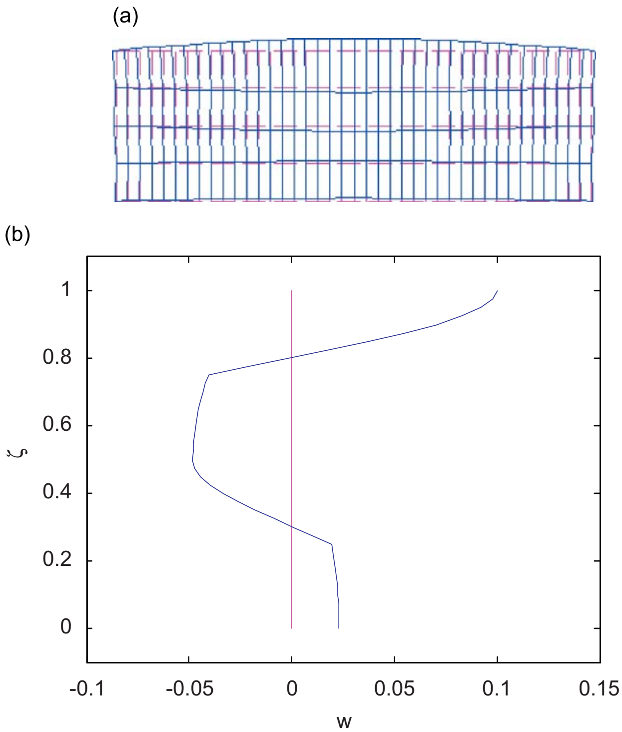

Fig. 7. The second thickness stretch mode ($\Omega = 5.29024761$) of the laminate [$Z/V/Z/V$] for $s = 0.1$, $n = 1$.

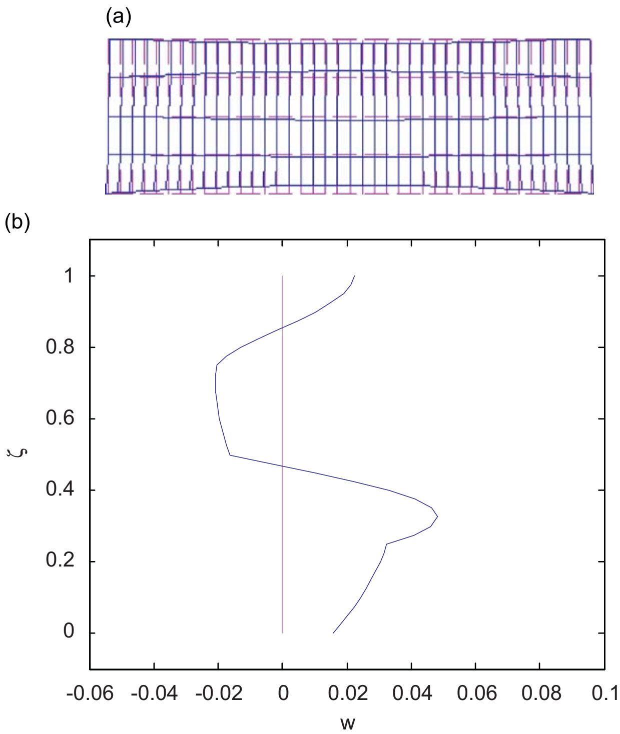

Fig. 8. The third thickness stretch mode ($\Omega = 7.29328146$) of the laminate [$Z/V/Z/V$] for $s = 0.1$, $n = 1$.

Figs. 9 and 10 show that surface wave modes usually dominate at high frequencies, which agrees with the well-known phenomenon of wave channeling in acoustics [36]. When the stacking sequence of layers changes, the channeling of vibration may occur at the lower surface or the interface of the laminate, as shown in Figs. 11 and 12 for the [$V/Z/V/Z$] and

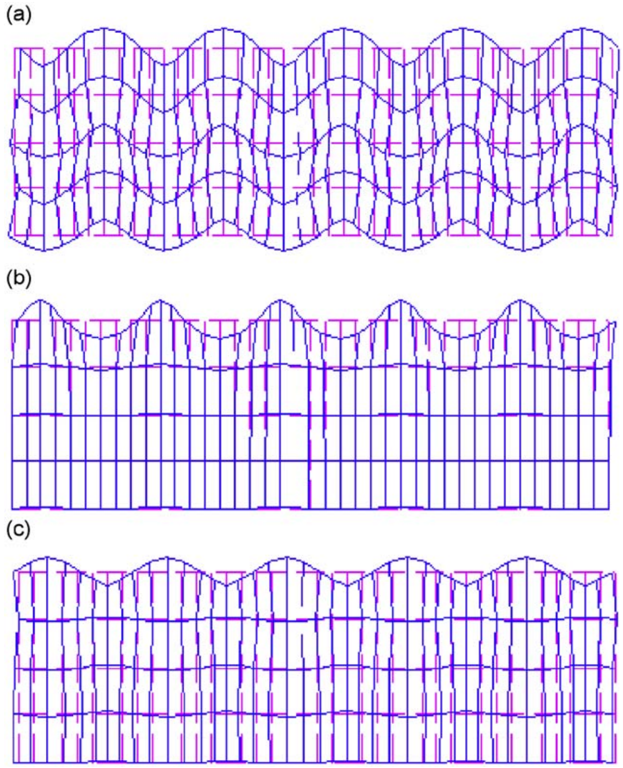

Fig. 9. Normalized mode of the laminate [Z/V/Z/V] for $s = 0.1$, $n = 10$: (a) $\Omega = 1.39341523$, (b) $\Omega = 3.46607641$ and (c) $\Omega = 4.42437516$.

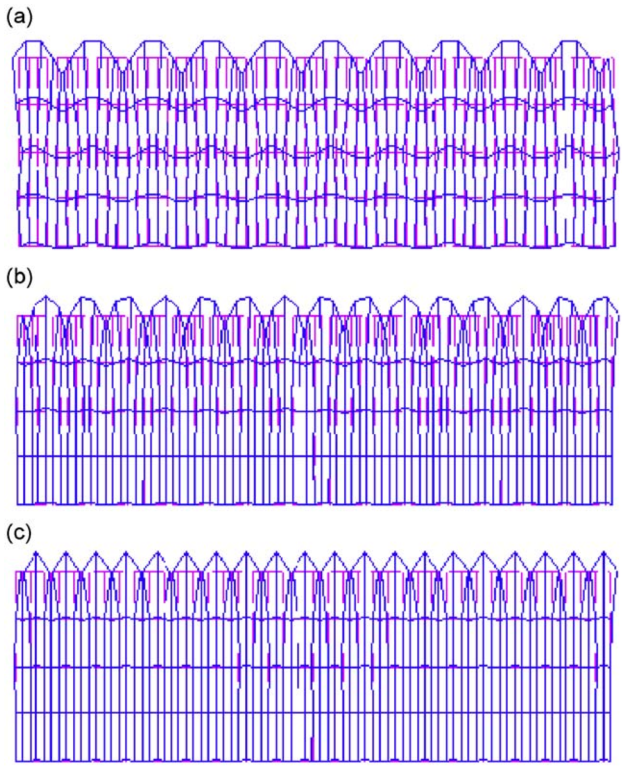

Fig. 10. Normalized mode of the laminate [Z/V/Z/V] for $s = 0.1$: (a) $n = 20$, $\Omega = 3.32573222$, (b) $n = 30$, $\Omega = 4.29974889$ and (c) $n = 40$, $\Omega = 4.95476922$.

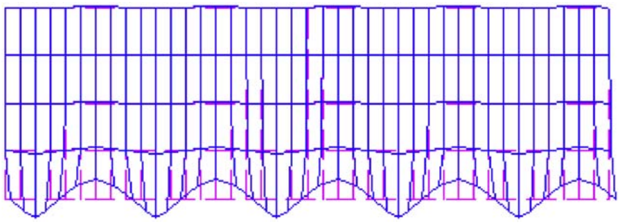

Fig. 11. Channeling of vibration at the lower surface of the laminate $[V/Z/V/Z]$ for $n = 10$, $s = 0.1$ at $\Omega = 11.96021219$.

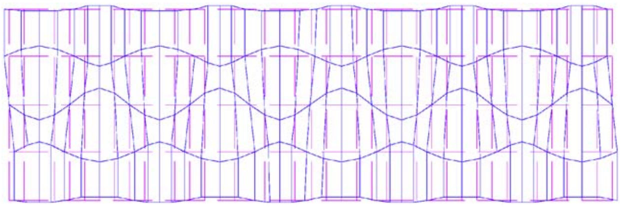

Fig. 12. Channeling of vibration at the intermediate interface of the laminate $[Z/V/V/Z]$ for $n = 10$, $s = 0.1$ at $\Omega = 6.32333149$.

$[Z/V/V/Z]$ laminates, respectively. In these cases, the energy is localized to the surface or interface and this property may be useful in nondestructive detection and evaluation of laminated composites.

7. Conclusion

This paper presents a theoretically solid basis for the method of reverberation-ray matrix using the theory of differential equations. The dual relation is derived based on the concept of coordinate transformation, while the joint coupling relation is derived from the compatibility and equilibrium conditions at the node or the interface/surface. Note that these results are obtained for a general linear differential equation system with constant coefficients as well as inhomogeneous terms.

Free vibration of cross-ply piezoelectric laminates in cylindrical bending is studied by employing the present modified MRRM. The coupling effect between elastic and electric fields is taken into consideration. Such factor seems to complicate the analysis, but based on the simple idea of MRRM, the derivation is still straightforward. In addition, numerical instabilities frequently encountered in the conventional SSM are removed by the present MRRM. The case of repeated eigenvalues is considered and the associated formulations are given. Numerical comparison with the SSM solution for simply supported laminate indicates that the MRRM is accurate and also efficient enough to deal with high frequency and long element problem.

Acknowledgments

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Appendix A

(1) Matrix operator **K**

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & 0 & \dfrac{1}{c_{55}} & -\dfrac{\partial}{\partial x} & -\dfrac{e_{15}}{c_{55}}\dfrac{\partial}{\partial x} \\ 0 & 0 & 0 & -\dfrac{\partial}{\partial x} & \rho\dfrac{\partial^2}{\partial t^2} & 0 \\ 0 & 0 & 0 & -\dfrac{e_{15}}{c_{55}}\dfrac{\partial}{\partial x} & 0 & \alpha\dfrac{\partial^2}{\partial x^2} \\ \rho\dfrac{\partial^2}{\partial t^2} + m_1\dfrac{\partial^2}{\partial x^2} & -\dfrac{m_2}{s}\dfrac{\partial}{\partial x} & \dfrac{m_3}{s}\sqrt{\dfrac{c_{55}^{(1)}}{\varepsilon_{33}^{(1)}}}\dfrac{\partial}{\partial x} & 0 & 0 & 0 \\ -\dfrac{m_2}{s}\dfrac{\partial}{\partial x} & \dfrac{m_4}{c_{55}^{(1)}} & \dfrac{m_5}{\sqrt{c_{55}^{(1)}\varepsilon_{33}^{(1)}}} & 0 & 0 & 0 \\ \dfrac{m_3}{s}\sqrt{\dfrac{c_{55}^{(1)}}{\varepsilon_{33}^{(1)}}}\dfrac{\partial}{\partial x} & \dfrac{m_5}{\sqrt{c_{55}^{(1)}\varepsilon_{33}^{(1)}}} & -\dfrac{m_6}{\varepsilon_{33}^{(1)}} & 0 & 0 & 0 \end{bmatrix} \tag{A.1}$$

(2) Induced variables σ_x and D_x

$$\sigma_x = -m_1 \frac{\partial u}{\partial x} + \frac{m_2}{s}\sigma_z - \frac{m_3}{s}\sqrt{\frac{c_{55}^{(1)}}{\varepsilon_{33}^{(1)}}} D_z, \quad D_x = \frac{e_{15}}{c_{55}}\tau_{xz} - \alpha\frac{\partial \phi}{\partial x} \tag{A.2}$$

(3) Coefficient matrix **C**

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & \dfrac{c_{55}^{(1)}}{c_{55}} & -n\pi s & -n\pi\beta \\ 0 & 0 & 0 & -n\pi s & \dfrac{\rho}{\rho^{(1)}}\Omega^2 & 0 \\ 0 & 0 & 0 & -n\pi\beta & 0 & n^2\pi^2 s^2 \dfrac{\alpha}{\varepsilon_{33}^{(1)}} \\ -\dfrac{\rho}{\rho^{(1)}}\Omega^2 - n^2\pi^2 s^2 \dfrac{m_1}{c_{55}^{(1)}} & n\pi m_2 & -n\pi m_3 & 0 & 0 & 0 \\ n\pi m_2 & -m_4 & -m_5 & 0 & 0 & 0 \\ -n\pi m_3 & -m_5 & m_6 & 0 & 0 & 0 \end{bmatrix} \tag{A.3}$$

(4) Some parameters

$$\alpha = \varepsilon_{11} + \frac{e_{15}^2}{c_{55}}, \quad \beta = \frac{e_{15}s}{c_{55}}\sqrt{\frac{c_{55}^{(1)}}{\varepsilon_{33}^{(1)}}}, \quad \Omega = h\omega\sqrt{\rho^{(1)}/c_{55}^{(1)}}$$

$$m_1 = \frac{c_{13}^2\varepsilon_{33} - c_{33}e_{31}^2 + 2c_{13}e_{31}e_{33}}{e_{33}^2 + c_{33}\varepsilon_{33}} - c_{11}, \quad m_2 = \frac{c_{13}\varepsilon_{33} + e_{31}e_{33}}{e_{33}^2 + c_{33}\varepsilon_{33}}s$$

$$m_3 = \frac{c_{33}e_{31} - c_{13}e_{33}}{e_{33}^2 + c_{33}\varepsilon_{33}}\sqrt{\frac{\varepsilon_{33}^{(1)}}{c_{55}^{(1)}}}s, \quad m_4 = \frac{\varepsilon_{33}c_{55}^{(1)}}{e_{33}^2 + c_{33}\varepsilon_{33}}, \quad m_5 = \frac{e_{33}\sqrt{c_{55}^{(1)}\varepsilon_{33}^{(1)}}}{e_{33}^2 + c_{33}\varepsilon_{33}}, \quad m_6 = \frac{c_{33}\varepsilon_{33}^{(1)}}{e_{33}^2 + c_{33}\varepsilon_{33}} \tag{A.4}$$

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